Harmonic maps on Riemannian foliations

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Abstract & Keyword

- **Abstract**

In this talk, we review harmonic maps between Riemannian foliations. There are two kinds of harmonic maps on Riemannian foliations, that is, transversally harmonic maps and $(\mathcal{F}, \mathcal{F}')$-harmonic maps. These are generalized to another harmonic maps, such as biharmonic maps, f-harmonic maps and F-harmonic map etc. We give recent works about these harmonic maps on foliations.

- **Keyword**

Riemannian foliation, transversal (bi-)energy, transversal (bi-)tension field, transversally (bi-)harmonic map, normal variation formulas.

Harmonic function

- Let (M, g) be a Riemannian manifold.
- A smooth function $f : M \to \mathbb{R}$ is **harmonic function** if

$$\Delta f = 0,$$

  where $\Delta f = \delta df$ and δ is the adjoint operator of d.
- Let $f : M \to \mathbb{R}^k$ be an isometric immersion. Then

$$\Delta f := (\Delta f_1, \cdots, \Delta f_k) = -nH,$$

  where H is the mean curvature vector field of M.
- An isometric immersion $f : M \to \mathbb{R}^k$ is harmonic if and only if $H = 0$ (that is, $f(M)$ is a minimal submanifold).
- **If M is closed, any harmonic function $f : M \to \mathbb{R}^k$ is constant.** So there does not exist a **closed minimal** submanifold on the Euclidean spaces.

Harmonic map

- A smooth map $\phi : (M, g) \to (N, h)$ is called **harmonic** if ϕ satisfies the Euler-Largrange equation $\tau(\phi) = 0$, where

$$\tau(\phi) := \mathrm{tr}_g \nabla d\phi$$

is called **tension field**.

- Note that for a function $f : M \to \mathbb{R}$,

$$\tau(f) = -\Delta f.$$

- Harmonic map is a generalization of harmonic function.

Biharmonic map

- (B.Y.Chen) $f : (M, g) \to \mathbb{R}^k$ is said to be biharmonic if

$$\Delta(\Delta f) = \Delta H = 0.$$

- A smooth map $\phi : (M, g) \to (N, h)$ is said to be **biharmonic** if $\tau_2(\phi) = 0$, where

$$\tau_2(\phi) = J^{\phi}(\tau(\phi)) := \nabla^*\nabla\tau(\phi) - \mathrm{tr}_g R^N(\tau(\phi), d\phi)d\phi$$

  is called **bitension field**. Here J^{ϕ} is the Jacobi operator.
- Trivially, any harmonic map is a biharmonic map.
- Biharmonic map is a generalization of biharmonic function. In fact, for a smooth function $f : M \to \mathbb{R}$,

$$\tau_2(f) = \nabla^*\nabla(\tau(f)) = -\nabla^*\nabla(\Delta f) = -\Delta(\Delta f).$$

- Generally, a biharmonic map is not harmonic.

Chen's conjectures

- (B.Y. Chen) If $\dim M = 2$, then any biharmonic submanifold of $\mathbb{R}^k$ is minimal.
- (Chen's conjecture, 1991) Any biharmonic submanifold of $\mathbb{R}^k$ must be minimal. Partially Yes (Akutagawa-Maeta, 2011).
- (Generalized Chen's conjecture, 2001) Every biharmonic submanifol M of a Riemannian manifold N with $K^N \leqslant 0$ must be harmonic (minimal) (by Caddeo-Montaldo-Piu). Partially Yes (Nakauchi-Urakawa (2013) under L^2-**condition of mean curvaure and completeness of** M).
- (Baird et al., 2010) If M is a non-compact **complete** manifold, $\mathrm{Ric}^M \geqslant 0$, $K^N \leqslant 0$ and $E_2(\phi) < \infty$, then a biharmonic map $\phi : M \to N$ is **harmonic**.
- (Nakauchi, Urakawa, Gudmundsson, 2014) If M is a non-compact **complete** manifold, $K^N \leqslant 0$ and $E(\phi) < \infty$, $E_2(\phi) < \infty$, then a biharmonic map $\phi : M \to N$ is **harmonic**. (cf. $\mathrm{Ric}^M \geqslant 0$ is not need)

Energy Functionals

- The **energy functional** of $\phi : M \to N$ is defined by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dM.$$

- The **bienergy functional** of $\phi : M \to N$ is defined by

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dM.$$

- The **variation formulas** are given by

$$\frac{d}{dt}E(\phi_t)|_{t=0} = -\int_M < \tau(\phi), V > dM,$$

$$\frac{d}{dt}E_2(\phi_t)|_{t=0} = -\int_M < \tau_2(\phi), V > dM,$$

where $V = \frac{d\phi_t}{dt}|_{t=0}$ is the variation vector fields of variations $\{\phi_t\}$ of ϕ.

Various harmonic maps

- Harmonic maps : $\tau(\phi) = 0 \iff$ a critical point of
$$E(\phi) = \frac{1}{2}\int |d\phi|^2.$$
- p-harmonic maps :
$\tau_p(\phi) = |d\phi|^{p-2}\tau(\phi) + (p-2)|d\phi|^{p-3}d\phi(\nabla|d\phi|) = 0$
$\iff$ a critical point of
$$E_p(\phi) = \frac{1}{p}\int |d\phi|^p, \ p > 1.$$
- f-harmonic maps : $\tau_f(\phi) = f\tau(\phi) + d\phi(\nabla f) = 0$
$\iff$ a critical point of
$$E_f(\phi) = \frac{1}{2}\int f|d\phi|^2, \quad f : M \to (0, \infty).$$

- Biharmonic maps : $\tau_2(\phi) = J^{\phi}(\tau) = 0 \iff$ cricital points of

$$E_2(\phi) = \frac{1}{2}\int |\tau(\phi)|^2.$$

- f-Biharmonic maps : $\tau_{2,f} := J^{\phi}(f\tau(\phi)) = 0 \iff$ critical points of

$$E_{2,f}(\phi) = \frac{1}{2}\int f|\tau(\phi)|^2.$$

- Bi-f-harmonic maps : $\tau_{2,f}(\phi) := fJ^{\phi}(\tau_f(\phi)) - \nabla_{\nabla f}\tau_f(\phi) = 0$ $\iff$ critical points of

$$E_{f,2}(\phi) = \frac{1}{2}\int |\tau_f(\phi)|^2, \quad f : M \to (0, \infty).$$

- Note that f-harmonic $\Rightarrow$ Bi-f-harmonic, but not f-biharmonic. Harmonic $\Rightarrow$ Biharmonic, f-biharmonic.

Liouville type theorems

- (Classical Liouville Theorem) Any bounded harmonic function defined on the whole plane (i.e. $\mathbb{R}^n$) must be constant.

- (Schoen-Yau, 1976) If M is complete and $\mathrm{Ric}^M \geqslant 0$ and $K^N \leqslant 0$, then any harmonic map $\phi : M \to N$ with $E(\phi) < \infty$ is constant.

- (Jung, 1997) Let M be complete with $\mathrm{Ric}^M \geqslant -\mu_0$ and N be of $K^N \leqslant 0$. Then any harmonic map $\phi : M \to N$ with $E(\phi) < \infty$ is constant.

- Here μ_0 is the infimum of the eigenvalues of the Laplacian acting on L^2-function on M.

Riemanian foliations

- Let $(M, g, \mathcal{F})$ be a Riemannian foliation with a bundle-like metric g.
- Let Q be the normal bundle of $\mathcal{F}$, that is, $Q = TM/T\mathcal{F}$. Roughly, Q is the tangent bundle of $M/\mathcal{F}$.
- Let g_Q be the holonomy invariant metric on Q induced by g.
- Let ∇ be the transversal Levi-Civita connection on Q.
- Let R^Q, K^Q, Ric^Q and σ^Q be the transversal curvature, sectional curvature, Ricci operator and scalar curvature, respectively.
- Let $\Omega_B^r(\mathcal{F})$ be a space of **basic form** ω, i.e., $i(X)\omega = 0, \quad i(X)d\omega = 0$ for any $X \in T\mathcal{F}$.
- Note that $\Omega^*(M) = \Omega_B^*(\mathcal{F}) \oplus \Omega_B^*(\mathcal{F})^\perp$.

Mean curvature form

- Let κ be the mean curvature form of $\mathcal{F}$ and κ_B is the basic part of κ.
- Trivially, $d\kappa_B = 0$.
- (Transversally divergence theorem) M, compact. Then for any $X \in \Gamma Q$,

$$\int_M \mathrm{div}_\nabla(X) = \int_M g_Q(\kappa, X).$$

- Let $d_B = d|_{\Omega_B}$ and δ_B: the adjoint operator of d_B, that is,

$$\delta_B \omega = (-1)^{q(r+1)+1} \bar{*}(d_B - \kappa_B \wedge)\bar{*}\omega,$$

  where $\bar{*}$ is the star operator on $\Omega_B^*(\mathcal{F})$.
- The **basic Laplacian** is defined by $\Delta_B = d_B\delta_B + \delta_B d_B$.

Transversally harmonic map

- Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map, that is, leaf preserving map, i.e., $d\phi(T\mathcal{F}) \subset T\mathcal{F}'$.
- Let $d_T\phi : Q \to Q'$ be defined by

$$d_T\phi := \pi' \circ d\phi \circ \sigma,$$

  and $\pi' : TM' \to Q'$ is a natural projection and $\sigma : Q \to T\mathcal{F}^\perp$ is an isomorphism.
- The **transversal tension field** of ϕ is defined by

$$\tau_b(\phi) := \mathrm{tr}_Q \nabla d_T\phi.$$

- (Konderak-Wolak, 2003) A smooth foliated map ϕ is said to be **transverally harmonic** if $\tau_b(\phi) = 0$.

$(\mathcal{F}, \mathcal{F}')$-harmonic map

- The **transversal energy** $E_B(\phi)$ of ϕ is defined by

$$E_B(\phi) = \frac{1}{2}\int_M |d_T\phi|^2.$$

- For a smooth foliated variation $\{\phi_t\}$,

$$\frac{d}{dt}E_B(\phi_t)|_{t=0} = -\int_M \langle V, \tilde{\tau}_b(\phi)\rangle,$$

  where

$$\tilde{\tau}_b(\phi) := \tau_b(\phi) - i(\kappa)d_T\phi.$$

- From the variation formula, it is trivial that a transversally harmonic map is not a critical point of $E_B(\phi)$.
- (Dragomir-Tommasoli (2013)) A **$(\mathcal{F}, \mathcal{F}')$-harmonic map** is a critical point of $E_B(\phi)$ $\iff$ a solution of $\tilde{\tau}_b(\phi) = 0$.

Remarks

- For a basic function $f : M \to \mathbb{R}$, we have

$$\tilde{\tau}_B(f) = -\Delta_B f.$$

- Hence $(\mathcal{F}, \mathcal{F}')$-harmonic map is a generalization of the basic harmonic function.

- Transversally harmonic map is a critical point of the special f-energy functional, where f is a solution of $d(\ln f) = \kappa_B$. (2013, Jung).

- When M is minimal, transversally harmonic map and $(\mathcal{F}, \mathcal{F}')$-harmonic map are equivalent.

Transversally biharmonic map

- The **transversal bitension field** $(\tau_2)_b(\phi)$ of ϕ is defined by

$$(\tau_2)_b(\phi) = J^T_\phi(\tau_b(\phi)),$$

where $J^T_\phi : \Gamma\phi^{-1}Q' \to \Gamma\phi^{-1}Q'$ is the transversal Jacobi operator defined by

$$J^T_\phi(V) = \nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}}V - \nabla_{\kappa^\sharp_B}V - \mathrm{tr}_Q R^{Q'}(V, d_T\phi)d_T\phi.$$

- (Chiang-Wloak, 2008) A foliated map $\phi : M \to M'$ is said to be **transversally biharmonic map** if $(\tau_2)_b(\phi) = 0$.

- Trivially, transversally harmonic map is transversally biharmonic.

Transversal bienergy functionals

- The **transversal bienergy functionals** of ϕ are defined by

$$(E_2)_B(\phi) = \frac{1}{2}\int_M |\tau_b(\phi)|^2$$

and

$$(\tilde{E}_2)_B(\phi) = \frac{1}{2}\int_M |\tilde{\tau}_b(\phi)|^2.$$

- **(Variation formulas for bienergy)** For a foliated map ϕ,

$$\frac{d}{dt}(E_2)_B(\phi_t)\Big|_{t=0} = -\int_M \langle (\tau_2)_b(\phi) + 2\nabla_{\kappa^\sharp}\tau_b(\phi), V\rangle$$

and

$$\frac{d}{dt}(\tilde{E}_2)_B(\phi_t)|_{t=0} = \int_M \langle (\tilde{\tau}_2)_b(\phi), V\rangle,$$

where the transveral bitension field is

$$(\tilde{\tau}_2)_b(\phi) := \nabla_{tr}^*\nabla_{tr}\tilde{\tau}_b(\phi) - tr_Q R^{Q'}(\tilde{\tau}_b(\phi), d_T\phi)d_T\phi.$$

$(\mathcal{F}, \mathcal{F}')$-biharmonic map

- Generally, the transversal biharmonic map is **not a critical point** of the transversal bienergy functional $(E_2)_B(\phi)$, but a critical point of the special f-bienergy functional with $d(\ln f) = \kappa_B$.
- $(\mathcal{F}, \mathcal{F}')$-biharmonic map is a **critical point of** $(\tilde{E}_2)_B(\phi)$ $\iff$ a solution of $(\tilde{\tau}_2)_b(\phi) = J_\phi(\tilde{\tau}_b(\phi)) = 0$, where the **generalized Jacobi operator** J_ϕ is defined by

$$J_\phi(V) := \nabla_{tr}^* \nabla_{tr} V - tr_Q R^{Q'}(V, d_T\phi)d_T\phi.$$

- $(\mathcal{F}, \mathcal{F}')$-harmonic map is trivially $(\mathcal{F}, \mathcal{F}')$-biharmonic. But the converse is not true.
- If ϕ is a basic function on M, that is, $\phi : M \to \mathbb{R}$, then

$$(\tilde{\tau}_2)(\phi) = J_\phi(\tilde{\tau}_b(\phi)) = J_\phi(\Delta_B\phi) = \nabla_{tr}^* \nabla_{tr}(\Delta_B\phi) = \Delta_B^2\phi.$$

- So $(\mathcal{F}, \mathcal{F}')$-biharmonic map is a generalization of biharmonic function.

Transversally f-harmonic map

- **Transversally f-harmonic map** is a solution of

$$\tau_{b,f}(\phi) := \mathrm{tr}_Q(\nabla_{tr} f d_T\phi) = f\tau_b(\phi) + d_T\phi(\nabla f) = 0$$

  for any nonzero basic function f.
- If f is constant, then transversally f-harmonic map is just transversally harmonic.
- The **transversal f-energy** of ϕ is defined by

$$E_{B,f}(\phi) = \frac{1}{2}\int_M f|d_T\phi|^2\mu_M.$$

- **(Variation formula for f-energy)**

$$\frac{d}{dt}E_{B,f}(\phi_t)|_{t=0} = -\int_M \langle V, \tau_{b,f}(\phi) - f d_T\phi(\kappa_B^\sharp)\rangle.$$

- Transversally f-harmonic map is not a critical point of the transversal f-energy $E_{B,f}(\phi)$.

$(\mathcal{F}, \mathcal{F}')_f$-harmonic map

- $(\mathcal{F}, \mathcal{F}')_f$**-harmonic map** is a critical point of tranversal f-energy functional $E_{B,f}(\phi) \iff$ a solution of

$$\tilde{\tau}_{b,f}(\phi) := \tau_{b,f}(\phi) - f d_T\phi(\kappa_B^\sharp) = 0.$$

- Let f_k be a solution of $\kappa_B = d(\ln f)$. Then $\tilde{\tau}_{b,f_\kappa}(\phi) = f_\kappa \tau_b(\phi)$.
- Then

$$\frac{d}{dt} E_{f_k}(\phi_t)|_{t=0} = -\int_M \langle V, \tau_b(\phi)\rangle f_k.$$

- **Any transversally harmonic map is a critical point of the transversal f_k-energy $E_{f_k}(\phi)$.**
- **Transversally harmonic map $\iff$ $(\mathcal{F}, \mathcal{F}')_{f_k}$-harmonic map.**

Transversally f-biharmonic map

- **Transversally f-biharmonic map** is a solution of

$$(\tau_2)_{b,f}(\phi) := J_\phi^T(f\tau_b(\phi)) = 0.$$

- The **transversal** f-**bienergy** of ϕ is given by

$$(E_2)_{b,f}(\phi) = \frac{1}{2}\int_M f|\tau_b(\phi)|^2.$$

- **(Variation formula for f-bienergy)**

$$\frac{d}{dt}(E_2)_{b,f}(\phi_t)\Big|_{t=0} = -\int_M \langle (\tau_2)_{b,f}(\phi) + 2\nabla_{\kappa_B^\sharp} f\tau_b(\phi), V\rangle$$

- So transversally f-biharmonic map is not a critical point of the transversal f-bienergy.

$(\mathcal{F}, \mathcal{F}')_f$-biharmonic map

- The **new transversal** f-**bienergy** of ϕ is given by

$$(\tilde{E}_2)_{b,f}(\phi) = \frac{1}{2}\int_M f|\tilde{\tau}_b(\phi)|^2,$$

where $\tilde{\tau}_b(\phi) = \tau_b(\phi) - d\phi(\kappa_B^\sharp)$.

- **(Variation formula for $(\tilde{E}_2)_{b,f}$)**

$$\frac{d}{dt}(\tilde{E}_2)_{b,f}(\phi_t)\Big|_{t=0} = -\int_M \langle (\tilde{\tau}_2)_{b,f}(\phi), V\rangle,$$

where

$$(\tilde{\tau}_2)_{b,f}(\phi) = J_\phi(f\tilde{\tau}_b(\phi)).$$

- $(\mathcal{F}, \mathcal{F}')_f$-**biharmonic map** is a critical point of the f-bienergy $(\tilde{E}_2)_{b,f}(\phi) \iff$ a solution of $(\tilde{\tau}_2)_{b,f}(\phi) = 0$.
- A $(\mathcal{F}, \mathcal{F}')_f$-harmonic map is not $(\mathcal{F}, \mathcal{F}')_f$-biharmonic map.

Transversally bi-f-harmonic maps

- Now, we define the **transversal** f-**Jacobi operator** $J^T_{\phi,f}$ by

$$J^T_{\phi,f}(V) := \nabla^*_{tr} f \nabla_{tr} V - f\nabla_{\kappa^\sharp} V - f\mathrm{tr}_Q R^{Q'}(V, d_T\phi)d_T\phi,$$

  where

$$\nabla^*_{tr} f \nabla_{tr} V = -\sum_a (\nabla_{E_a} f \nabla_{E_a} V - f\nabla_{\nabla_{E_a} E_a} V) + f\nabla_{\kappa^\sharp} V.$$

- **Transversally bi-f-harmonic map** is a solution of the folowing equation

$$(\tau_{b,f})_2(\phi) := J^T_{\phi,f}(\tau_{b,f}) = 0.$$

- **Note that Chiang-Wolak called as "transversally f-biharmonic map" in their paper (2013).**
- Firstly, Ouakkas-Nasri-Djaa called "f-biharmonic map" (2010). Lu changed the terminology to "bi-f-harmonic map"(2013).

Transversal bi-f-energy

- Now, consider the transversal bi-f-energy functional $(E_{B,f})_2(\phi)$ as

$$(E_{B,f})_2(\phi) := \frac{1}{2}\int_M |\tau_{b,f}(\phi)|^2 \mu_M.$$

- (Variation formula)

$$\frac{d}{dt}(E_{B,f}(\phi_t)|_{t=0} = -\int \langle V, J^T_{\phi,f}(\tau_{b,f}) + 2f\nabla_\kappa \tau_{b,f} + \kappa(f)\tau_{b,f}\rangle \mu_M.$$

- Transversally bi-f-harmonic map is not a critical point of $(E_{B,f})_2(\phi)$.

Bi-$(\mathcal{F},\mathcal{F}')_f$-harmonic map

- We consider the transversal bi-f-energy functional as

$$(\tilde{E}_{b,f})_2(\phi) := \frac{1}{2}\int |\tilde{\tau}_{b,f}(\phi)|^2.$$

- Bi-$(\mathcal{F},\mathcal{F}')_f$-harmonic map is a critical point of the energy functional $(\tilde{E}_{b,f})_2(\phi)$.
- The generalized Jacobi operator $J_{\phi,f}$ is given by

$$J_{\phi,f}(V) := J^T_{\phi,f} + f\nabla_{\kappa^\sharp}V.$$

  Then $J_{\phi,f}$ is a formally self adjoint operator.
- (Variation formula)

$$\frac{d}{dt}(\tilde{E}_{B,f})_2(\phi_t)_{t=0} = -\int_M \langle J_{\phi,f}(\tilde{\tau}_{b,f}), V\rangle.$$

- A bi-$(\mathcal{F},\mathcal{F}')_f$-harmonic map is a solution of $(\tilde{\tau}_{b,f})_2 = 0$, where

$$(\tilde{\tau}_{b,f})_2(\phi) := J_{\phi,f}(\tilde{\tau}_{b,f}(\phi)).$$

Remarks

- Transversally harmonic maps $(\tau_b = 0)$
  $\subset$ Transversally biharmonic maps $(J^T_\phi(\tau_b) = 0)$;

- $(\mathcal{F}, \mathcal{F}')$-harmonic maps $(\tilde{\tau}_b = 0)$
  $\subset$ $(\mathcal{F}, \mathcal{F}')$-biharmonic maps $(J_\phi(\tilde{\tau}_b) = 0)$.

- Transversally f-harmonic maps $(\tau_{b,f} = 0)$
  $\subset$ Transversally bi-f-harmonic maps $(J^T_{\phi,f}(\tau_{b,f}) = 0)$;
  But $\not\subseteq$ Transversally f-biharmonic maps $(J^T_\phi(f\tau_b) = 0)$.

- $(\mathcal{F}, \mathcal{F}')_f$-harmonic maps $(\tilde{\tau}_{b,f} = 0)$
  $\subset$ Bi-$(\mathcal{F}, \mathcal{F}')_f$-harmonic maps $(J_{\phi,f}(\tilde{\tau}_{b,f}) = 0)$;
  but $\not\subseteq$ $(\mathcal{F}, \mathcal{F}')_f$-biharmonic maps $(J_\phi(f\tilde{\tau}_b) = 0)$.

Transversally p-harmonic map

- The transversally p-harmonic map is a solution of $\tau_{b,p}(\phi) = 0$, where

$$\tau_{b,p}(\phi) := \mathrm{tr}_Q(\nabla_{tr}(|d_T\phi|^{p-2}d_T\phi) \\ = |d_T\phi|^{p-2}\{\tau_b(\phi) + (p-2)d_T\phi(\nabla_{tr}\ln|d_T\phi|)\}.$$

- Transversally harmonic map is a transversally 2-harmonic map.
- The transversal p-energy functional of ϕ is given by

$$E_{B,p}(\phi) = \frac{1}{p}\int |d_T\phi|^p.$$

- (Variation formula) For a smooth foliated map ϕ,

$$\frac{d}{dt}E_{B,p}(\phi_t)|_{t=0} = -\int\langle\tilde{\tau}_{b,p}(\phi), V\rangle,$$

where $\tilde{\tau}_{b,p}(\phi) = \tau_{b,p}(\phi) - |d_T\phi|^{p-2}d_T\phi(\kappa_B^\sharp)$.

$(\mathcal{F}, \mathcal{F}')_p$ -harmonic map

- **So transversally p-harmonic map is not a critical point of $E_{B,p}(\phi)$.**

- The **$(\mathcal{F}, \mathcal{F}')_p$-harmonic map** is a **critical point** of $E_{B,p}(\phi)$ $\iff$ a solution of $\tilde{\tau}_{b,p}(\phi) = 0$.

- Trivially, $(\mathcal{F}, \mathcal{F}')_2$-harmonic map is a $(\mathcal{F}, \mathcal{F}')$-harmonic map.

F-harmonic maps

- Let $F : [0, \infty) \to [0, \infty)$ be a C^2-function with $F' > 0$. Then ϕ is transversally F-harmonic if ϕ is a solution of $\tau_{b,F}(\phi) = 0$, where

$$\tau_{b,F}(\phi) = F'(\frac{|d_T\phi|^2}{2})\tau_b(\phi) + d_T\phi\Big(\nabla_{tr}F'(\frac{|d_T\phi|^2}{2})\Big).$$

- When $F(s) = s$, the transversal F-tension field $\tau_{b,F}(\phi)$ is the transversal tension field $\tau_b(\phi)$.
- Any transversally F-harmonic map ϕ without critical points is a transversally f-harmonic map with $f = -\ln F'(\frac{|d_T\phi|^2}{2})$.
- A map ϕ is said to be $(\mathcal{F}, \mathcal{F}')_F$-harmonic map if it is a critical point of the functional

$$E_{B,F}(\phi) = \int_M F\Big(\frac{|d_T\phi|^2}{2}\Big).$$

Weitzenbock formulas

- Let $\Omega_B^r(E)$ be the space of E-valued basic r-forms, where $E = \phi^{-1}Q'$. Then $d_\nabla : \Omega_B^r(E) \to \Omega_B^{r+1}(E)$ is defined by

$$d_\nabla(\omega \otimes s) = (-1)^r \omega \wedge \nabla s + d_B \omega \otimes s.$$

- Let δ_∇ be the formal adjoint of d_∇ and the Laplacian $\Delta = d_\nabla \delta_\nabla + \delta_\nabla d_\nabla$.
- Trivially, for a smooth foliated map ϕ,

$$d_\nabla(d_T\phi) = 0, \quad \delta_\nabla d_T\phi = -\tau_b(\phi) + i(\kappa_B^\sharp)d_T\phi.$$

- If ϕ is transversally harmonic, then

$$\delta_\nabla(d_T\phi) = i(\kappa_B^\sharp)d_T\phi.$$

- If ϕ is $(\mathcal{F}, \mathcal{F}')$-harmonic, then

$$\delta_\nabla(d_T\phi) = \tilde{\tau}_b(\phi) = 0.$$

- (Weitzenbock formula)

$$\frac{1}{2}\Delta_B|d_T\phi|^2 =< \Delta d_T\phi, d_T\phi > -|\nabla_{tr} d_T\phi|^2 - < F(d_T\phi), d_T\phi > - < A_{\kappa^\sharp} d_T\phi, d_T\phi >,$$

where $A_Y s = L_Y s - \nabla_Y s$ for any $Y \in \Gamma Q$ and

$$< F(d_T\phi), d_T\phi >= \sum_a < d_T\phi(Ric^Q(E_a)), d_T\phi(E_a) > - \sum_{a,b} K^{Q'}(d_T\phi(E_a), d_T\phi(E_b)).$$

- If ϕ is transversally harmonic, then

$$\frac{1}{2}(\Delta_B - \kappa_B^\sharp)|d_T\phi|^2 = -|\nabla_{tr} d_T\phi|^2 - < F(d_T\phi), d_T\phi > .$$

- If ϕ is $(\mathcal{F}, \mathcal{F}')$-harmonic, then

$$\frac{1}{2}\Delta_B|d_T\phi|^2 = -|\nabla_{tr} d_T\phi|^2 - < F(d_T\phi), d_T\phi > \\ - < d_\nabla i(\kappa_B^\sharp) d_T\phi, d_T\phi > + \frac{1}{2}\kappa_B^\sharp(|d_T\phi|^2).$$

- Note that

$$d_\nabla(|d_T\phi|^{p-2} d_T\phi) = d_B(|d_T\phi|^{p-2}) \wedge d_T\phi,$$
$$\delta_\nabla(|d_T\phi|^{p-2} d_T\phi) = -\tau_{b,p}(\phi) + |d_T\phi|^{p-2} i(\kappa_B^\sharp) d_T\phi.$$

- If ϕ is transvesally p-harmonic, then

$$\delta_\nabla(|d_T\phi|^{p-2}) = |d_T\phi|^{p-2} i(\kappa_B^\sharp) d_T\phi.$$

- If ϕ is $(\mathcal{F}, \mathcal{F}')_p$-harmonic, then

$$\delta_\nabla(|d_T\phi\phi|^{p-2}) = -\tilde{\tau}_{b,p}(\phi) = 0.$$

Generalized Chen's conjecture on foliations

- Let $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a transversally biharmonic map. Then ϕ is transversally harmonic under one of the following conditions:

  (1) (Chiang-Wolak, 2008) M closed and $K^{Q'} \leqslant 0$;

  (2) (Jung-Jung, 2017) M complete, $\mathrm{Vol}(M) = \infty$, all leaves are compact, $K^{Q'} \leqslant 0$ and $(E_2)_B(\phi) < \infty$;

  (3) (Ohno-Sakai-Urakawa, 2017) M complete, $K^{Q'} \leqslant 0$, $\mathrm{div}_\nabla(S_T(\phi)) = 0$, $(E_2)_B(\phi) < \infty$ and $(E_2)_B(\phi) < \infty$.

- Let $\phi : M \to M'$ be a $(\mathcal{F}, \mathcal{F}')$-biharmonic map with M closed. Assume that $K^{Q'} \leqslant 0$. Then ϕ is $(\mathcal{F}, \mathcal{F}')$-harmonic. (In preparation).

Liouville type theorems on foliations

- (Jung-Jung, 2012) Let M is compact, $\mathrm{Ric}^Q \geqslant 0$ and $K^{Q'} \leqslant 0$. Then any transversally harmonic map is transversally totally geodesic, that is, $\nabla_{tr} d_T \phi = 0$. In addition, if there exists a point such that $\mathrm{Ric}^Q > 0$, then ϕ is transversally constant.
- (Jung - Jung, 2017) Let $(M, \mathcal{F})$ be complete with infinite volume and $\mathrm{Ric}^Q \geqslant 0$ and $K^{Q'} \leqslant 0$. If all leaves are compact, then every transversally harmonic map $\phi : M \to M'$ of $E_B(\phi) < \infty$ is transversally constant.
- (Fu-Jung, 2021) Let $(M, \mathcal{F})$ be complete with $\mathrm{Ric}^Q \geqslant -\mu_0$ and $> -\mu_0$ at some point and $K^{Q'} \leqslant 0$. If all leaves are compact, then every transversally harmonic map $\phi : M \to M'$ of $E_B(\phi) < \infty$ is transversally constant.

- (Fu-Jung, 2023) Let M be a complete and all leaves are compact. Assume that $\mathrm{Ric}^Q \geqslant -\frac{4(p-1)}{p^2}\mu_0 (p \geqslant 2)$ and > 0 at some point and $K^{Q'} \leqslant 0$. Then any $(\mathcal{F}, \mathcal{F}')_p$-harmonic map with $E_{B,p}(\phi) < \infty$ is transversally constant.

- **We do not know whether that holds for the transversally p-harmonic map $(p > 2)$. When $p = 2$, the above theorem holds (Fu-Jung, 2021).**
- **Generalized Chen's conjectures and Liouville type theorems for f-harmonic, F-harmonic and p-harmonic maps on foliated manifolds ???**

Thank you for your attention!